

Optimal Switching Frequency in Limited-Cycle with Multiple Periods

Jing Sun*

Graduate School of Engineering, Nagoya Institute of Technology

Hisashi Yamamoto

Department of System Design, Management Systems Engineering, Tokyo Metropolitan University

Masayuki Matsui

Department of Informatics Graduate School of Informatics and Engineering,

The University of Electro-Communications

Xianda Kong

Department of System Design, Management Systems Engineering, Tokyo Metropolitan University

(Received: December 7, 2011 / Revised: February 17, 2012 / Accepted: February 18, 2012)

ABSTRACT

Due to the customer needs of reducing cost and delivery date shorting, prompt change in the production plan became more important. In the multi period system (For instance, production line.) where target processing time exists, production, idle and delay risks occur repeatedly for multiple periods. In such situations, delay of one process may influence the delivery date of an entire process. In this paper, we discuss the minimum expected cost of the case mentioned above, where the risk depends on the previous situation and occurs repeatedly for multiple periods. This paper considers the optimal switching frequency to minimize the total expected cost of the production process. In this paper, first, the optimal switching frequency model is proposed. Next, the mathematic formulation of the total expectation is presented. Finally, the policy of optimal switching frequency is investigated by numerical experiments.

Keywords: Limited-Cycle Problem, Multi-Period, Optimal Switching Frequency, Production Seat System, Manufacturing Line

* Corresponding Author, E-mail: sun.jing@nitech.ac.jp

1. INTRODUCTION

Due to the customer needs of reducing cost and delivery date shorting, prompt change in the production plan became more important.

In the social system or production process of multiple periods (or machines or workers) where target processing time exists, idle and delay risks occur repeatedly for multiple periods. In such situations, delay of one process may influence the delivery date of an entire process. In this paper, we discuss the minimum expected cost of the case mentioned above, where the risk depends on the previous situation and occurs repeatedly for multiple

periods. There exists an object with some constraints (e.g., due date of production process). These constraints produce a risk and the object occurs repeatedly for multiple periods. This kind of problem is called "a limited-cycle problem with multiple periods" (Yamamoto *et al.*, 2006), and is seen in production lines, time-bucket balancing, production seat systems and so on (Matsui, 2008).

Under the condition of uncertainty, the result or efficiency of a period is often controlled not only by the risks of this period but also by the risks generated beforehand. Whether the process (period or seat) satisfies the due time (restriction) usually depends on the state of the process beforehand, as seen in Wright *et al.* (1974),

Verzijl *et al.* (1976) and Benders *et al.* (2002).

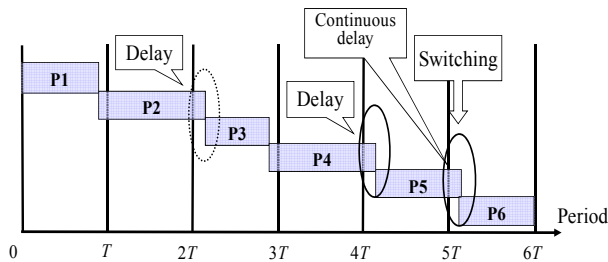


Figure 1. Optimal Switching Frequency Problem
 ($n = 6, k = 2$).

In particular, in the case of the risk which depends on a situation of a past process (for instance, the case of a production line for a multi period), how to assign machines, workers or jobs to be most efficient and economical is a problem (optimal assignment) in load/risk planning (for example, see Bergamashi *et al.*, 1997 and Swamidass *et al.*, 2000).

Yamamoto *et al.* (2006, 2007, 2010a, 2010b) and Sun *et al.* (2010) considered the optimal arrangement to minimize the total expected cost of such a situation. Before describing a limited-cycle problem with multiple periods, we consider a single limited-cycle problem. If the production time of one period is T and the due time is Z , then the risks due to the length of the production time would occur, which are the risks by $T \leq Z$ and $T > Z$. In such situations, it can be noted that there is a trade-off problem in the two risks. This problem is shown as

$$\min_z \{ \beta_1(Z)P_r(T \leq Z) + \beta_2(Z)P_r(T > Z) \},$$

where $\beta_1(Z)$ is the risk by $T \leq Z$, and $\beta_2(Z)$ is the risk by $T > Z$. This kind of problem is known as various problems of the reliability field, the problem of due time restriction and the newsboy problem (a well-known stock problem) (Week, 1979). A detailed model of a single limited-cycle is shown in Matsui (2005).

This paper considers cases in which the above two risks not only occur in the single period, but also in multiple periods repeatedly. The problem of minimizing the expected cost in such a situation is a limited-cycle problem with multiple periods. The multi period problem could be classified according to whether the periods are independent or not

For this problem, one result is the general form of production rate and waiting time by a station-centered approach as discussed in Matsui *et al.* (1977) and Matsui (2004). The explicit form is obvious and consists of the product form in the period-independent case, such as a single line, but it is untraceable in the period-dependent case such as a mixed or tandem line. The mixed line has a absorbing barrier, but the tandem line has a reflective barrier at the end. This paper presents a cost

approach for the latter.

This paper considers the optimal switching frequency to minimize the total expected cost of the production process. In this paper, first, the optimal switching frequency model is proposed. Next, the mathematic formulation of the total expectation is presented. Finally, the policy of optimal switching frequency is investigated by numerical experiments.

2. OPTIMAL SWITCHING MODEL

In the production process of multi periods, delay of one process may influence the delivery date of an entire process. We consider controlling the production process by switching the processing rate to a faster one at a given point. The optimal switching problem is when the processing rate should be switched to minimize the total expectation cost. In this paper, we consider the optimal switching point to minimize the total expected cost of the production process. In our considered model, if processing are delay k times continuously, the processing rates should be changed at the next period. Figure 1 shows an example of a switching time model in multiple periods, where period number n is 6 and switching point is 2 (delay continuous times).

2.1 The Assumption and Notation

The optimal switching frequency model for the multi limited-cycle is considered based on the following assumptions:

- (1) One product is made by a process with n processes.
- (2) For $i = 1, 2, \dots, n$, the production time of process i is denoted by T_i which is assumed to be exponentially distributed and statistically independent, respectively. The usual processing rate is μ_1 , and the emergency processing rate is μ_2 .
- (3) For $i = 1, 2, \dots, n$, the target production time of process i is denoted by T , which is set by standard of the process, and the due time of the entire process (n periods) is nT .
- (4) k is continuous delay times.
- (5) The cost per unit time ($C_s^{(i)}$) occurs when a process is executed before the target production time of the process ($i = 1$ means before switching and $i = 2$ means after switching).
- (6) The cost per unit time ($C_p^{(i)}$) occurs when a process is executed after the target production time of the process ($i = 1$ means before switching and $i = 2$ means after switching).
- (7) When $X_i = \sum_{l=1}^i T_l > \text{due time } (nT)$ of n periods, the delay cost $C_p^{(i)}$ occurs.
- (8) When $X_i = \sum_{l=1}^i T_l < \text{due time } (nT)$ of n periods, the idle cost C_s occurs.

Some notations are also defined.

For $i = 1, 2, \dots, n$,

$C(T_1, T_2, \dots, T_n)$: the total cost of the production process.

$C(i)$: the production cost of period i .

T_i : the production time of period i .

X_i : the production time of i periods ($X_i = \sum_{l=1}^i T_l$).

$\Pr(X_n > nT)$: the probability of delay.

$\Pr(X_n \leq nT)$: the probability of idle.

2.2 Problem Formulation

By assumptions mentioned in section 2.1, the object function is set as follows:

$$E[C(T_1, T_2, \dots, T_n)] = \sum_{i=1}^n E[C(i; T_1, T_2, \dots, T_i)] + C_p P_r \{X_n > nT\} + C_s P_r \{X_n \leq nT\} + E[C(n+1)] \quad (1)$$

where,

$E[C(i; T_1, T_2, \dots, T_i)]$ is the expected cost of period i ,

$C_p P_r \{X_n > nT\}$ is the delayed expected cost,

$C_s P_r \{X_n \leq nT\}$ is the idle expected cost,

$E[C(n+1)]$ is production cost after due time,

and for $i = 1, 2, \dots, n$

$$E[C(i; T_1, T_2, \dots, T_i)] = \left. \begin{aligned} & C_s^{(1)} \left\{ \begin{aligned} & \iint_{0 \leq x_{i-1} + t_i \leq iT} t_i f_1(t_i) g^c(i-1; x_{i-1}) dt_i dx_{i-1} \\ & + \iint_{x_{i-1} \leq iT \leq x_{i-1} + t_i} (iT - x_{i-1}) f_1(t_i) g^c(i-1; x_{i-1}) dt_i dx_{i-1} \end{aligned} \right\} \\ & + C_p^{(1)} \left\{ \begin{aligned} & \iint_{iT \leq x_{i-1} + t_i \leq \infty} t_i f_1(t_i) g^c(i-1; x_{i-1}) dt_i dx_{i-1} \\ & + \iint_{x_{i-1} \leq iT \leq x_{i-1} + t_i} (x_{i-1} + t_i - iT) f_1(t_i) g^c(i-1; x_{i-1}) dt_i dx_{i-1} \end{aligned} \right\} \\ & + C_s^{(2)} \left\{ \begin{aligned} & \iint_{0 \leq x_{i-1} + t_i \leq iT} t_i f_2(t_i) g(i-1; x_{i-1}) dt_i dx_{i-1} \\ & + \iint_{x_{i-1} \leq iT \leq x_{i-1} + t_i} (iT - x_{i-1}) f_2(t_i) g(i-1; x_{i-1}) dt_i dx_{i-1} \end{aligned} \right\} \\ & + C_p^{(2)} \left\{ \begin{aligned} & \iint_{iT \leq x_{i-1} + t_i \leq \infty} t_i f_2(t_i) g(i-1; x_{i-1}) dt_i dx_{i-1} \\ & + \iint_{x_{i-1} \leq iT \leq x_{i-1} + t_i} (x_{i-1} + t_i - iT) f_2(t_i) g(i-1; x_{i-1}) dt_i dx_{i-1} \end{aligned} \right\} \end{aligned} \right. \quad (2)$$

where x_{i-1} is the total time from start to period $(i-1)$ ending.

In this paper, for $i = 1, 2, \dots, n$, the production time of process i is denoted by T_i which is assumed to be exponentially distributed and statistically independent, respectively. The usual processing rate is μ_1 , and the emergency processing rate is μ_2 .

Based on the relation between exponential distribu-

tion and Poisson distribution, the terms in equation (1) are given as follows:

For $i = 1, 2, \dots, n$ and $t \geq 0$,

$G^c(i; t)$: The distribution function of total production costs of period i ended on time t , and process doesn't exceed k times continuously from period 1 to i (before switching).

$g^c(i; t)$: The probability density function of $G^c(i; t)$.

$G(i; t)$: The distribution function of total production costs of period i ended on time t , and process doesn't exceed k times continuously from period 1 to i (after switching).

$g(i; t)$: The probability density function of $G(i; t)$.

$G_e^c(i; t)$: The probability of period i ended on time t , and process doesn't exceed k times continuously from period 1 to i , and $X_i < t$.

$G_e(i; t)$: The probability of period i ended on time t , and process exceed k times continuously from period 1 to i , and $X_i < t$.

$H_e(i, l; t)$: The probability of period i ended on time t , process doesn't exceed k times continuously from period 1 to $i-l-1$, and period $i-l$ doesn't exceed, and period $i-l-1$ to i all exceed.

Theorem 1:

For $i = 1, 2, \dots, n$,

$$E[C(i)] = \sum_{l_1=1}^n \sum_{l_2=l_1}^n EC(i; l_1, l_2) + \sum_{l=1}^n EC^{(0)}(i; l) \quad (3)$$

Theorem 2:

For $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, n$,

$$EC^{(0)}(i; l) = \begin{cases} G_e^c(l-1; (i-1)T) C_s^{(1)} T e^{-\mu_1 T} \\ + G_e(l-1; (i-1)T) C_s^{(2)} T e^{-\mu_2 T} & \text{if } l \geq i \\ G_e^c(l-1; (i-1)T) C_p^{(1)} T e^{-\mu_1 T} \\ + G_e(l-1; (i-1)T) C_p^{(2)} T e^{-\mu_2 T} & \text{if } l < i \end{cases} \quad (4)$$

$EC^{(0)}(i; l)$ is the expected production cost of l period from $(i-1)$ period to i period under the production process conditions shown in Figure 2.

Theorem 3:

For $i = 1, 2, \dots, n$

$$E[C(i)] = \sum_{l_1=1}^{i-1} \sum_{l_2=l_1}^n \sum_{y=0}^{k-1} H_e(l_1-1, y; (i-1)T) EC_T^{(1,2)}(i; l_1, l_2, l_1+k-y-1) + \sum_{l_1=1}^n \sum_{l_2=l_1}^n G_e^c(l_1-1; (i-1)T) EC^{(0)}(i; l_1, l_2) + \sum_{l_1=1}^n \sum_{l_2=l_1}^n G_e(l_1-1; (i-1)T) EC^{(2)}(i; l_1, l_2) \quad (5)$$

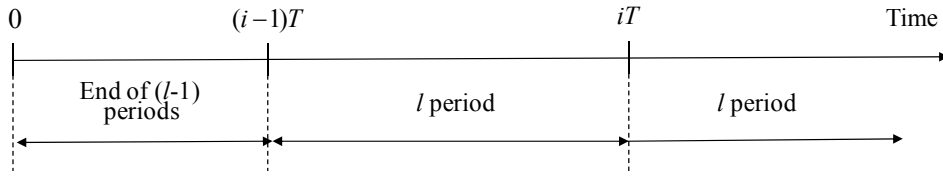


Figure 2. Definition of $EC^{(0)}(i; l)$.

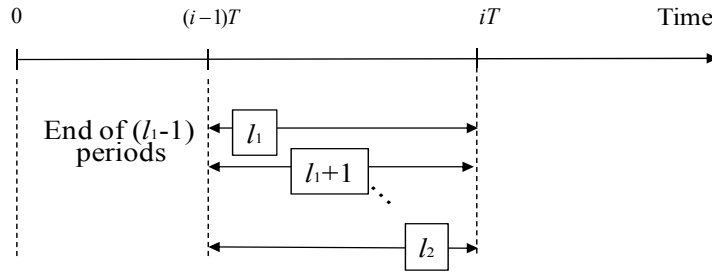


Figure 3. Definition of $EC^{(1)}(i; l_1, l_2)$.

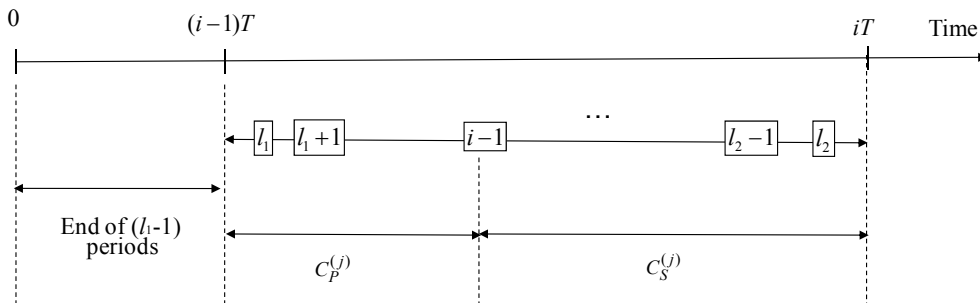


Figure 4. Definition of $EC^{(2)}(i; l_1, l_2)$.

$$\begin{aligned}
 & + \sum_{l=1}^{i-1} [G_e^c(l-1; (i-1)T) C_p^{(1)} T e^{-\mu_1 T} + G_e(l-1; (i-1)T) C_p^{(2)} T e^{-\mu_2 T}] \\
 & + \sum_{l=1}^n [G_e^c(l-1; (i-1)T) C_s^{(1)} T e^{-\mu_1 T} + G_e(l-1; (i-1)T) C_s^{(2)} T e^{-\mu_2 T}]
 \end{aligned}$$

$EC_T^{(1,2)}(i; l_1, l_2, l_1 + k - y - 1)$ is defined in (iii) of Theorem 4, where $l_1 + k - y - 1$ is a switching point, and $0 < y \leq l_1 + k - n - 2$.

Theorem 4:

- (i) For $i = 1, 2, \dots, n$, $l_1 = i, i + 1, \dots, n$ and $l_2 = l_2, l_2 + 1, \dots, n$ ($i \leq l_1, l_1 \leq l_2$),

$$EC^{(1)}(i; l_1, l_2) = L_1^{(1)}(l_2) \cdot T \cdot \frac{(\mu_1 T)^{l_2 - l_1 + 1}}{(l_2 - l_1 + 1)!} e^{-\mu_1 T} \quad (6)$$

$EC^{(1)}(i; l_1, l_2)$ is the expected production cost of l period from $(i-1)$ period to i period under the production process conditions shown in Figure 3.

- (ii) For $i = 1, 2, \dots, n$, $l_1 = 1, 2, \dots, n$, and $l_2 = l_1, l_1 + 1, \dots, n$ ($l_1 \leq l_2$),

$$EC^{(2)}(i; l_1, l_2) = \begin{cases} L_1^{(2)}(l_2) \cdot T \cdot \frac{(\mu_2 T)^{l_2 - l_1 + 1}}{(l_2 - l_1 + 1)!} e^{-\mu_2 T} & \text{if } i \leq l_1 \\ \frac{T}{l_2 - l_1 + 2} [C_p^{(2)}(i - l_1) + L_2^{(2)}(l_2)] \frac{(\mu_2 T)^{l_2 - l_1 + 1}}{(l_2 - l_1 + 1)!} e^{-\mu_2 T} & \text{if } l_1 \leq i - 1 < l_2 \\ \frac{T}{l_2 - l_1 + 2} [C_p^{(2)}(l_2 - l_1 + 1) + L_3^{(2)}(l_2)] \cdot \frac{(\mu_2 T)^{l_2 - l_1 + 1}}{(l_2 - l_1 + 1)!} e^{-\mu_2 T} & \text{if } l_2 \leq i - 1 \end{cases} \quad (7)$$

$EC^{(2)}(i; l_1, l_2)$ is the expected production cost of l period from $(i-1)$ period to i period under the production process conditions shown in Figure 4.

- (iii) For $i = 1, 2, \dots, n - 1$, $l_1 = 1, 2, \dots, i - 1$, $l_2 = l_1, l_1 + 1, \dots, n$ ($i > l_1, l_1 \leq l_2$) and $l = 1, 2, \dots, l = 1, 2, \dots, k + (l_1 - 1)$ ($l - (l_1 - 1) \leq k$),

$$EC_T^{(1,2)}(i; l_1, l_2, l) = \begin{cases} \frac{T}{l_2 - l_1 + 2} \left[C_p^{(1)}(i - l_1) + C_s^{(1)}(l_2 - i + 1) + L_4(l_2) \right] \cdot \frac{(\mu_1 T)^{l_2 - l_1 + 1}}{(l_2 - l_1 + 1)!} e^{-\mu_1 T} & \text{if } l > l_2, l_1 \leq i - 1 \text{ or } l \geq i, l_1 \leq i - 1 \\ \int_0^T L_5(l_2; t) \frac{(\mu_1 t)^{l_2 - l_1}}{(l_2 - l_1)!} e^{-\mu_1 t} \frac{(\mu_2 (T - t))^{l_2 - l_1}}{(l_2 - l_1)!} e^{-\mu_2 (T - t)} \mu_1 dt & \text{if } l < i, l \leq l_2, l_1 \leq i - 1 \end{cases} \quad (8)$$

where, for $j = 1, 2$,

$$L_1^{(j)}(l_2) = \begin{cases} C_s^{(j)} & \text{if } l_2 < n \\ C_s^{(j)} \frac{l_2 - l_1 + 1}{l_2 - l_1 + 2} & \text{if } l_2 = n \end{cases} \quad (9)$$

and

$$L_2^{(2)}(l_2) = \begin{cases} C_s^{(2)}(l_2 - i + 2) & \text{if } l_2 < n \\ C_s^{(2)}(l_2 - i + 1) & \text{if } l_2 = n \end{cases} \quad (10)$$

$$L_3^{(2)}(l_2) = \begin{cases} C_p^{(2)} & \text{if } l_2 < i - 1 \\ C_s^{(2)} & \text{if } l_2 = i - 1 \end{cases} \quad (11)$$

$$L_4(l_2) = \begin{cases} C_s^{(2)} & \text{if } l_2 < n \\ 0 & \text{if } l_2 = n \end{cases} \quad (12)$$

$$L_5(l_2; t) = \begin{cases} C_p^{(1)}t + C_p^{(2)} \frac{i - l - 1}{l_2 - l + 1} (T - t) + C_s^{(2)} \frac{l_2 - i + 2}{l_2 - l + 1} (T - t) & \text{if } l_2 < n \\ C_p^{(1)}t + C_p^{(2)} \frac{i - l - 1}{l_2 - l + 1} (T - t) + C_s^{(2)} \frac{l_2 - i + 1}{l_2 - l + 1} (T - t) & \text{if } l_2 = n \end{cases} \quad (13)$$

Theorem 5:

For $i = 1, 2, \dots, n - 1, l_1 = 1, 2, \dots, i - 1$,

$$E[C(n+1)] = \sum_{l_1=1}^n G_e(l_1 - 1; nT) \cdot C_p^{(2)} \left[\frac{1}{\mu_2} \cdot (n - l_1 + 1) \right] + \sum_{l_1=1}^n \sum_{y=0}^{l_1 + k - n - 2} \left\{ H_e(l_1 - 1, y; nT) \cdot C_p^{(1)} \left[\frac{1}{\mu_1} \cdot (n - l_1 + 1) \right] \right\} + \sum_{l_1=1}^n \sum_{y=\max(0, l_1 + k - n - 1)}^{k-1} \left\{ H_e(l_1 - 1, y; nT) \cdot \left(C_p^{(1)} \left[\frac{1}{\mu_1} \cdot (k - y) \right] + C_p^{(2)} \left[\frac{(n - l_1 - k + y + 2)}{\mu_2} \right] \right) \right\} \quad (14)$$

3. EXPERIMENTAL CONSIDERATION

In this section, we consider the optimal switching times to minimize the total expected cost by numerical

experiments, where $\mu_2 = 0.3, C_s = 10, \mu_1 = 0.1, C_s^{(1)} = 4, T = 3, C_p^{(1)} = 6, C_s^{(2)} = 8, C_p^{(2)} = 10, C_p = 20$.

We show three cases by production process situations as follows:

- (I) Total cost concludes production cost before due time, delay and idle costs.
- (II) Total cost concludes production cost (before due time and after due time), delay and idle costs.
- (III) Total cost concludes production cost (before due time and after due time).

We can note that (II) is the case of equation (1).

Figure 5 and Figure 6 show the relation of expected costs and switching point k (i.e. continuous delay times).

Figure 5 shows the behavior of the optimal switching point by change of continuous delay times (k) when $n = 5$. From Figure 5, it can be noted that when cost cases are I, II and III, the optimal switching k are 1, 4 and 4, respectively.

Figure 6 shows the behavior of the optimal switching point by change of continuous delay times (k) when $n = 10$. From Figure 6, it can be noted that when cost cases are I, II and III, the optimal switching k are 1, 2 and 2, respectively.

From Figure 5 and Figure 6, it also can be noted that optimal switching point k is smaller when the period number n is large. It means the rate should be switched to a faster one ahead of time when the period number is large. This is because the delay cost could be decreased.

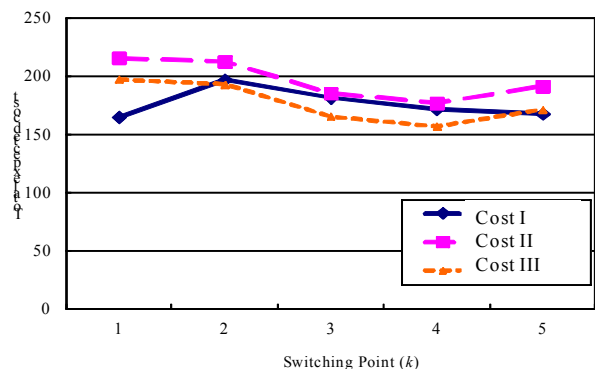


Figure 5. Behavior of the Optimal Switching ($n = 5$).

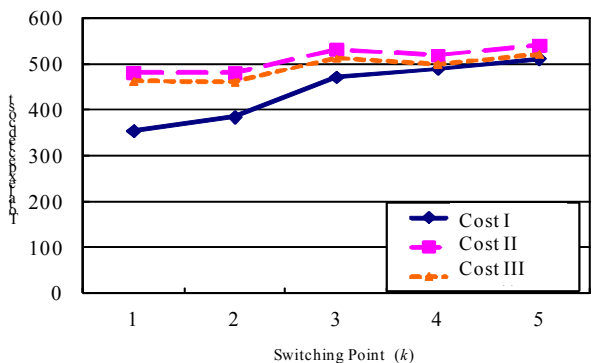


Figure 6. Behavior of the Optimal Switching ($n = 10$).

4. CONCLUSION

In this paper, we considered the optimal switching point to minimize the total expected cost in limited-cycle problems with multiple periods. First, the optimal switching frequency model is proposed. Next, the mathematic formulation of the total expectation is presented. Finally, the policy of optimal switching frequency is investigated by numerical experiments, and the optimal switching point could be found, too.

ACKNOWLEDGMENT

This research was partially supported by the Ministry of Education, Science, Sports and Culture, a Grant-in-Aid for Scientific Research(C), 23510161 in 2011.

REFERENCES

- Benders, J. (2002), The origin of period batch control (PBC), *Int. J. Prod. Res.*, **40**, 1-6.
- Bergamashi, D., Cigolin, R., Perona, M., and Portiol, A. (1997), Order review and release strategies in job shop environment: a review and a classification, *Int. J. Prod. Res.*, **35**, 399-420.
- Matsui, M., Shingu, T., and Makabe, H. (1977), Conveyor-serviced production system: an analytic framework for station-centered approach by some queuing formulas, *Preliminary Report of the Operations Research Society of Japan*, 104-107 (in Japanese).
- Matsui, M. (2004), MGM-modeling and balancing problem in supply chain management, *Proceedings of APIEMS*, Gold Coast, Australia (on CD-ROM).
- Matsui, M. (2005), A Management cycle model: switching control under lot processing and time span, *Journal of Japan Industrial Management Association*, **56**, 256-264.
- Matsui, M. (2008), *Manufacturing and Service Enterprise with Risk: A Stochastic Management Approach*, Springer.
- Sun J., Yamamoto H., and Matsui M. (2010), A study of optimal switching problem in limited-cycle with multiple periods, *Journal of Zhejiang University-SCIENCE A (Applied Physics and Engineering)*, **11**, 933-937.
- Swamidass, P. M. (2000), *Encyclopedia of Production and Manufacturing Management*, Kluwer, Boston.
- Verzija, J. J. (1976), *Production Planning and Information Systems*, Macmillan, London, 16-17.
- Week, J. K. (1979), Optimizing planned lead times and delivery dates, *21st Annual Conference Proceedings, APICS*, 177-188.
- Wright, Q. W. (1974), *Production and Inventory Management in the Computer Age*, Canners Publishing Co.
- Yamamoto, H., Matsui M. and Liu J. (2006), A basic study on a limited-cycle problem with multi periods and the optimal assignment problem, *Journal of Japan Industrial Management Association*, **57**, 23-31 (in Japanese).
- Yamamoto, H., Matsui M. and Bai X. S. (2007), A Branch and Bound Method for the Optimal Assignment during a Limit-cycle Problem with Multiple Periods, *Journal of Japan Industrial Management Association*, **58**(37-43) (in Japanese).
- Yamamoto, H., Sun, J., Oishi, T., and Matsui, M. (2010a), A Study of the Optimal Switching Point of the Limited-Cycle Problem with Multiple Periods, *Journal of Japan Industrial Management Association*, **61**(4) (in Japanese).
- Yamamoto, H., Sun, J., and Matsui, M. (2010b), A Study on Limited-Cycle Scheduling Problem with Multiple Periods, *Computer and Industrial Engineering*, **59**(44), 675-681.